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Joint optimization of maintenance policy and inspection interval for a multi-unit series system using proportional hazards model

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Unlike the previous maintenance models of multi-unit systems which considered condition-based maintenance (CBM) or age information separately, we propose a novel optimization model which is characterized by a combination of CBM and age information using proportional hazards model. The preventive maintenance is applied for the main two units, where one unit is the core part of the system and subject to CM, and only the age information for the second main unit is available. Also, the other units are adjusted or replaced each time when the system is maintained. The objective is to find an optimal opportunistic maintenance policy minimizing the long-run expected average cost per unit time. The problem is formulated and solved in the semi-Markov decision process framework. The formula for the mean residual life of the system is derived, which is an important statistic in practical applications. A practical example of a multi-unit system from a mining company is provided, and a comparison with other policies shows an outstanding performance of the new model and the control policy proposed in this paper.

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1. Introduction

Production facilities are subject to deterioration and failure as the result of usage and age, which considerably reduce the efficiency of the production systems. To overcome this deficiency, different maintenance models have been developed depending on various assumptions such as historical data availability, optimization criterion, and perfect or imperfect maintenance actions.

Among different approaches, CBM has been applied widely in various industries depending on the collected CM data, including oil data in Wang and Hussin (2009) and Kim *et al* (2011) or vibration data in Yam *et al* (2001) and Tian *et al* (2014). In a CBM program, maintenance action is chosen based on the information collected through CM (Jardine *et al*, 2006). Recently, Rosmaini and Shahrul (2012) reviewed and compared CBM and time-based maintenance (TBM). They investigated the challenges of implementing each technique from a practical point of view and concluded that the application of CBM is more realistic and more worthwhile. CBM models are classified into two main categories (Si *et al*, 2011): (1) *Directly Observed CBM Models*, such as regression-based models, Markovian-based models, or Gamma processes describing system deterioration and (2) *Indirectly Observed CBM Models*, e.g., proportional hazards models with imperfect information and hidden Markov or semi-Markov models. The existing literature has formed primarily on determining the optimal CBM policy for single-unit systems, while development of CBM for multi-unit systems is in its infancy.

An extensive review on maintenance modeling and optimization of multi-unit systems has been presented by Thomas (1986), Cho and Parlar (1991), Dekker *et al* (1997), Wang (2002), Nicolai and Dekker (2006) and Nowakowski and Werbinka (2009). Interaction among different working units of a multi-unit system is an important factor when developing maintenance policies which differentiate the problems encountered in these models from the problems when dealing with single-unit systems. In the early maintenance literature (Thomas, 1986), three types of interactions have been introduced: (1) economic dependence, (2) structural dependence, and (3) stochastic dependence. In this paper, we assume that the multi-unit series system has economic dependence.

To develop an effective maintenance model, CBM is one of the appropriate approaches; however, the age of the unit should not be neglected. To overcome this deficiency, Cox (1972) introduced the PHM in 1972. An extensive literature review on PHM also can be found in Kumar and Klefsjo (1994). PHM has gained popularity as a useful model for different applications, such as steel and mining industry (Zuashkiani *et al*, 2009), oil and petrochemical industry (Vlok *et al*, 2002; Makis *et al*, 2006). Since there are usually several

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covariates in real applications, Lin *et al* (2006) proposed an approach using principal component analysis instead of the original covariates to build the PHM and got reasonable results by reducing the number of variables included in the model. Also, Makis *et al* (2006) applied the dynamic principle component analysis to take into account both the cross- and autocorrelation to reduce the number of covariates in the PHM.

Maintenance problems in multi-unit systems are considerably more complicated when compared to single-unit systems maintenance policies. Such systems are more applicable to real-world situations, but there has been very limited research especially using PH model. For example, Zheng and Fard (1991) studied the combination of age-based and opportunistic maintenance for a multi-unit system consisting of n identical units with increasing hazard rates. A unit is replaced at failure or at a predetermined age, whichever occurs first. The other units are replaced opportunistically if their ages are within the specified limits. The optimal limits are obtained by minimizing the mean total replacement cost rate. Another example can be found in Tian and Liao (2011), where the authors built a PH model for a multi-unit system and they supposed that there is economic dependence among units. They introduced two thresholds d_1 and d_2. If the hazard rate of unit 1 crosses the first threshold, PM action will be performed. To take advantage of the economic dependency among different units, the second threshold will be used. If the hazards functions of the other units exceed d_2 then opportunistic maintenance will be performed on those units as well. More related papers for multi-unit systems using PHM can be found in Marseguerra *et al* (2002), Barata *et al* (2002), Castanier *et al* (2005), Saunil *et al* (2009), and Koochaki *et al* (2012).

In this paper, we propose a joint optimization of maintenance policy and inspection interval for a multi-unit series system with economic dependence. One unit is the core part of the system, so it is subject to CM and PHM is used to describe the hazard function of this unit. The second main unit is considered to be less critical; however, its failure may cause substantial damage to the system; therefore, its age information is available. The other units are adjusted or simply replaced upon performing maintenance on the two main units. The distinguishing feature of the proposed model is that we utilize the combination of CM and age information in maintenance decision-making for a multi-unit series system using PHM. This is the first contribution of this paper, which has not been considered before and it is applicable in many real situations. The motivation for choosing this assumption in our modeling is that in many real systems, spending money on CM for all the units is not economical and reasonable. Companies are interested in spending less money on non-value-added activities such as CM and simultaneously minimize interruptions caused by system failure. This model is more realistic than the one-unit models, and substantial savings can be achieved by applying the model in various situations, such as considering steam generator as a critical unit in a nuclear power plant, gearbox in the hauler trucks, or crude oil export pump on an oil rig, the cracker in an oil refinery, or gearbox in the wind turbine (Kumar and Jain 2012). For instance, a gearbox of the heavy hauler truck is considered as a main unit, subject to CM. The second important unit can be a clutch for which just the age information is available, and its failure can make considerable damage to the system. When system maintenance is performed, the other units such as bearings, engine belt, radial shaft seal, and rings can be replaced or adjusted. This kind of application will be considered later in experimental results section using real data.

Surprisingly, little research has been done on the combined CM and age-based models of multi-unit systems, which appears to be a good representation of real systems. A related model for a one-unit system was developed, for example in Makis and Jardine (1992) which presented the optimal maintenance policy for a PH model minimizing the long-run expected average cost per unit time. They considered a PHM with a Markov covariate process and periodic monitoring. Later, Wu and Ryan (2010) extended their work by considering possible state transitions between sampling epochs. We have further extended the assumption of transitions in the covariate process made in Wu and Ryan (2010) by relaxing the sequential degradation from state i to $i+1$ considered in their paper, to a general type of degradation.

Another contribution of the paper is using the SMDP framework to obtain the optimal maintenance policy which is again a novel approach to maintenance modeling of multi-unit series systems using PHM. Such a maintenance policy has a direct practical value as it can be readily implemented for online decision-making. The decision maker can decide when the CM information should be collected, as well as when to initiate preventive and opportunistic maintenance.

The remainder of the paper is organized as follows. The details of the proposed model are summarized in model formulation section. Then, a computational algorithm in the SMDP framework based on the policy iteration algorithm is developed. The derivation and computation of the mean residual life are presented in residual life prediction section. The effectiveness of the proposed model is demonstrated by using a practical example from a mining company in experimental results section. Finally, we discuss possible extensions of our model and provide concluding remarks.

2. Model formulation

Consider a system consisting of N operating units with two main units or modules. One main unit (unit 1) is the core part of the system, it is assumed to be more expensive than the other units, and it is subject to CM. Only the age information of another main unit (unit 2) is available, and $(N-2)$ remaining units are cheaper units which can be adjusted or replaced easily when the system is maintained. There is an economic dependence among these units, i.e., upon performing jointly maintenance actions, economies of scales are incorporated.

To describe the behavior of the unit 1 deterioration process properly, the value of the covariate process is determined through inspections. This CM information and the age of the unit are incorporated into the PH model. The hazard rate is the product of a baseline hazard rate $h_0(t)$ dependent on the age of unit 1 and a positive function $\psi(Z_t)$ dependent only on the values of the covariate process. Let Z be a continuous-time Markov chain with the state space $\Omega = \{0, 1, 2, \ldots, J\}$, where some subsets of states represent healthy and warning (unhealthy) conditions, and the last state (J) is an absorbing state. The mathematical derivations are based on these coded values. We also assume that there is no deterioration at time zero, i.e., $Z_0 = 0$. Thus, the hazard rate at time t can be expressed as follows:

$$h(t, Z_t) = h_0(t)\psi(Z_t), \tag{1}$$

and the survival function is given by

$$P(\xi_1 > t | Z_s, 0 \le s \le t) = \exp\Big(- \int_0^{\mathrm{t}} \mathrm{h_0(s)}\psi(\mathrm{Z_s})\mathrm{ds}\Big), \tag{2}$$

where ξ_1 is the failure time of unit 1.

Only age information of the second main unit (unit 2) is available, and its lifetime distribution is of a general type denoted by $f_2(t)$, and ξ_2 represents its failure time. Therefore, the failure time of the system is denoted by $\xi = min(\xi_1, \xi_2)$.

Unit 1 covariate process values (Z) are known only at discrete time epochs ($\Delta, 2\Delta, \ldots, n\Delta$), where Δ is the inspection interval. Using Eq. (1), the hazard rate at the n^{th} inspection epoch is given by:

$$h(n\Delta, Z_{n\Delta}) = h_0(n\Delta)\psi(Z_{n\Delta}). \tag{3}$$

The Z process is a continuous-time Markov chain, and its instantaneous transition rates $q_{ij}, i,j \in \Omega$ are defined by

$$\begin{aligned} q_{ij} &= \lim_{u \to 0^+} \frac{P(Z_{t+u} = j | Z_t = i)}{u} < +\infty, \quad i \ne j \in \Omega \\ q_{ii} &= -\sum_{i \ne j} q_{ij}. \end{aligned} \tag{4}$$

To model monotonic system deterioration, we assume that the state process is non-decreasing with probability 1, i.e., $q_{ij} = 0$ for all $j < i$.

We will show a detailed development for 3 states, which can be further extended to a larger number of states. We assume two operational states and one absorbing state for the covariate process. In most practical applications (see, for example, Kim *et al*, 2011), considering only two operational states is sufficient for fault detection and CBM. The first phase is the normal or healthy phase where the measurements of the covariate process obtained from CM behave approximately as a stationary process. However, when the degradation exceeds certain level, the behavior of the CM measurements changes substantially.

For our Markov covariate process with 3 states, the transition rate matrix is given by:

$$\mathbf{Q} = \begin{bmatrix} -(q_{01} + q_{02}) & q_{01} & q_{02} \\ 0 & -q_{12} & q_{12} \\ 0 & 0 & 0 \end{bmatrix}. \tag{5}$$

where $q_{ij} = v_i P_{ij}$, for $i \ne j$. The transition probability matrix $\mathbf{P}(t) = (P_{ij}(t))_{i,j \in \Omega}$ is obtained by solving the Kolmogorov backward differential equations (Tijm, 1994), where

$$\begin{aligned} \mathbf{P}(t) &= [P_{ij}(t)] \\ &= \begin{bmatrix} e^{-v_0 t} & \dfrac{q_{01}(e^{-v_1 t} - e^{-v_0 t})}{v_0 - v_1} & 1 - e^{-v_0 t} - \dfrac{q_{01}(e^{-v_1 t} - e^{-v_0 t})}{v_0 - v_1} \\ 0 & e^{-v_1 t} & 1 - e^{-v_1 t} \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{6}$$

After collecting the covariate value Z at each inspection epoch and processing the information, the hazard rate of unit 1 is obtained using Eq. (3), and then, proper action is taken. If the hazard rate does not cross the preventive maintenance level (U), then unit 1 is left operational without any intervention until the next inspection epoch. Once the hazard rate exceeds the preventive maintenance level (U) or when the age of unit 1 exceeds the predetermined maximum useful operating age T_1, then all units will be opportunistically replaced. Unit 1 failure can occur at any time, and upon its failure, failure replacement of unit 1 and preventive replacements of all the other units are performed. The second main unit is preventively maintained considering the age-based replacement policy. When unit 2 fails, or it is preventively replaced at the optimal maintenance time T_2, unit 1 hazard rate is updated at these times. If the updated hazard rate exceeds the opportunistic maintenance level (W), then unit 1 is opportunistically maintained; otherwise, it is left operational. We consider the following cost components in the model:

- C_I: Inspection cost incurred whenever we take an inspection.
- C_{P1}: Preventive maintenance cost of unit 1, which takes T_P time units.
- C_{F1}: Corrective maintenance cost of unit 1, which takes T_F time units.
- C_{P2}: Preventive maintenance cost of unit 2.
- C_{F2}: Replacement cost of unit 2.
- C_{OP}: Adjustment cost of $(N - 2)$ units.
- C_{LP}: Cost rate related to the loss of production incurred when the production is stopped to perform preventive, or corrective maintenance.
- C_s: Set-up cost incurred whenever the system is stopped.

The objective is to find the optimal value of the opportunistic and preventive maintenance levels (W^*, U^*), preventive maintenance time of unit 2 (T_2), as well as the inspection interval Δ^* such that the long-run expected average cost per unit time is minimized. In the next section, we develop an efficient computational algorithm in the semi-Markov decision process (SMDP) framework to determine the optimal decision variables $(W^*, U^*, T_2^*, \Delta^*)$.

3. Computational algorithm in the SMDP framework

In this section, we develop the computational algorithm in the SMDP framework. We start monitoring unit 1 at equidistant inspection epochs. Suppose that at inspection time $n\Delta$, unit 1 is operational, i.e., $\xi_1 > n\Delta$. Then, we compute unit 1 hazard rate $h(n\Delta, Z_{n\Delta})$ using Eq. (3). We partition the hazard rate interval [0, H] into L subintervals, where H is a suitably selected upper bound for the hazard rate. The number of subintervals (L) should also be selected properly. The larger number of subintervals causes more accurate results; however, it also increases the computational time, so it should not be selected very small to get sufficiently precise results in a reasonable time.

Now, the definition of the state space in the SMDP is required. We define the set $S_1 = \{(0,0,0)\}$ when both units are new or "as good as new," and $S_2 = \{(z,n,r) \mid z \in \Omega, n, r \in N, h(n\Delta, Z_{n\Delta}) < U\}$, where the first component indicates the value of the covariate at time $n\Delta$ and the second and third components represent the age of unit 1 and unit 2, respectively, i.e., ($n\Delta$) and ($r\Delta$), when unit 1 hazard rate is below the preventive maintenance level. If the hazard rate of unit 1 exceeds the preventive maintenance level (U), the SMDP is defined to be in state $PM2$, and we define the set $S_3 = \{PM_2\}$. Similarly, when unit 2 failure occurs or upon its preventive maintenance time (T_2), then the hazard rate of unit 1 is updated. If the updated hazard rate crosses the opportunistic maintenance level, then the SMDP is defined to be in state $PM1$, where $S_4 = \{PM_1\}$.

Thus, the state space of the SMDP is given by $S = S_1 \cup S_2 \cup S_3 \cup S_4$. Now, the following quantities should be determined to obtain the optimal long-run expected average cost (Tijm, 1994):

$P_{m,k}$ = the probability that unit 1 hazard rate will be in state k given the current state is m, where $m, k \in S$.

τ_m = the expected sojourn time until the next decision epoch given the current state is $m \in S$.

C_m = the expected cost incurred until the next decision epoch given the current state is $m \in S$.

Once all these quantities are defined, for a fixed preventive maintenance level U, opportunistic maintenance level W, unit 2 preventive maintenance time T_2, and inspection interval Δ, the long-run expected average cost $g(W, U, T_2, \Delta)$ can be obtained by solving the following system of linear equations (Tijm, 1994):

$$V_m = C_m - g(W, U, T_2, \Delta)\tau_m + \sum_{k \in S} P_{m,k} V_k, \quad \text{for } m \in S$$
$$V_l = 0, \quad \text{for an arbitrarily selected state } l \in S. \tag{7}$$

So, the optimal decision variables $(W^*, U^*, T_2^*, \Delta^*)$ and the corresponding minimum long-run expected average cost per unit time $g(W^*, U^*, T_2^*, \Delta^*)$ can be found by iteratively solving Eq. (7).

To derive the closed-form expressions for the SMDP quantities, the calculation of the conditional reliability function is required. The conditional reliability function of unit 1 can be obtained by:

$$\overline{R}_1(n,z,t) = P(\xi_1 > n\Delta + t \mid \xi_1 > n\Delta, Z_{1\Delta}, \ldots, Z_{n\Delta} = z). \tag{8}$$

Since the degradation state process is only observable at each inspection epoch, it may transit at any time between two inspection epochs (Wu and Ryan, 2010). We suppose that covariate process can make transition from healthy state to other states, whereas Wu and Ryan (2010) considered the sequential degradation, i.e., the covariate process can only make transitions from state i to $i+1$. Therefore, conditional reliability function can be rewritten as follows:

$$\overline{R}_1(n,z,t) = E\left[\exp\left(-\int_{n\Delta}^{n\Delta+t} h_0(s)\psi(Z_s)ds\right) \middle| Z_{n\Delta} = z\right]. \tag{9}$$

The above equation can be evaluated based on the different values of z and conditioning on the covariate process sojourn times in the healthy and unhealthy states (see "Appendix 1").

3.1. Transition probabilities

This section is devoted to the derivation of the transition probabilities for the system states.

1. Assume that the system is in the state $(z, (n-1), (r-1))$, where $h((n-1)\Delta, z) < U$, $n\Delta < T_1$, and $r\Delta < T_2$. Then, the transition probability to the state (z', n, r) where $h(n\Delta, z') < U$ and unit 2 works properly, is given by:

$$P_{(z,(n-1),(r-1)),(z',n,r)} = P(Z_{n\Delta} = z', \xi_1 > n\Delta, \xi_2 > r\Delta \mid n\Delta < T_1, r\Delta < T_2, \xi_1 > (n-1)\Delta, \xi_2 > (r-1)\Delta, Z_{(n-1)\Delta} = z). \tag{10}$$

   It is the probability that the value of the hazard rate will not exceed the preventive maintenance level U and the system will not fail in the next inspection interval. Then, this probability can be calculated as follows:

$$P_{(z,(n-1),(r-1)),(z',n,r)} = P(Z_{n\Delta} = z', \xi_1 > n\Delta, \xi_2 > r\Delta \mid \cdot P(\xi_1 > n\Delta | Z_{n\Delta} = z', \xi_1 > (n-1)\Delta) \cdot \frac{R_2(r\Delta)}{R_2((r-1)\Delta)} = P_{z,z'}(\Delta) \cdot \overline{R}_1((n-1), z, \Delta) \cdot \frac{R_2(r\Delta)}{R_2((r-1)\Delta)}, \tag{11}$$

where the first term can be derived from Eq.(6) and the second and third terms are reliability functions of unit 1 and unit 2, respectively.

2. If the hazard rate crosses the preventive maintenance level $h(n\Delta, z') \geq U$, or the age of unit 1 exceeds the predetermined age T_1, where $r\Delta < T_2$, then the system goes to $PM2$ state and the corresponding transition probability is given by:

$$P_{(z,(n-1),(r-1)),(PM2)} = \begin{cases} \sum_{z'} P(Z_{n\Delta}=z'|\xi_1 > n\Delta, \xi_1 > (n-1)\Delta, Z_{(n-1)\Delta}=z) \\ \cdot \overline{R}_1((n-1),z,\Delta) \cdot \frac{R_2(r\Delta)}{R_2((r-1)\Delta)}; & n\Delta < T_1 \\ \overline{R}_1((n-1),z,\Delta) \cdot \frac{R_2(r\Delta)}{R_2((r-1)\Delta)}; & n\Delta \geq T_1. \end{cases} = \begin{cases} \sum_{z'} P_{z,z'}(\Delta) \cdot \overline{R}_1((n-1),z,\Delta) \cdot \frac{R_2(r\Delta)}{R_2((r-1)\Delta)}; & n\Delta < T_1 \\ \overline{R}_1((n-1),z,\Delta) \cdot \frac{R_2(r\Delta)}{R_2((r-1)\Delta)}; & n\Delta \geq T_1. \end{cases} \tag{12}$$

where z' are the covariate values that cause the hazard rate to exceed the preventive maintenance level.

3. When the age of unit 2 exceeds its preventive maintenance time T_2, and if the hazard rate of unit 1 crosses the opportunistic maintenance level W, then all units are preventively replaced, and we have:

$$P_{(z,(n-1),(r-1)),(PM1)} = P(PM1, \xi_1 > n\Delta, \xi_2 > r\Delta \mid n\Delta < T_1, r\Delta \geq T_2, \xi_1 > (n-1)\Delta, \xi_2 > (r-1)\Delta, Z_{(n-1)\Delta} = z) = \sum_{z'} P_{z,z'}(\Delta) \cdot \overline{R}_1((n-1), z, t) \cdot \frac{R_2(r\Delta)}{R_2((r-1)\Delta)} \tag{13}$$

where z' are the covariate values that cause the hazard rate to exceed the opportunistic maintenance level. Otherwise, unit 1 is left operational, and the rest of units are preventively maintained. The corresponding transition probability is obtained as below:

$$P_{(z,(n-1),(r-1)),(z',n,1)} = P(Z_{n\Delta} = z', \xi_1 > n\Delta, \xi_2 > r\Delta \mid n\Delta < T_1, r\Delta \geq T_2, \xi_1 > (n-1)\Delta, \xi_2 > (r-1)\Delta, Z_{(n-1)\Delta} = z) = P_{z,z'}(\Delta) \cdot \overline{R}_1((n-1), z, t) \cdot \frac{R_2(r\Delta)}{R_2((r-1)\Delta)} \tag{14}$$

where z' is the covariate value that causes the hazard rate not to exceed the opportunistic maintenance level.

4. When unit 2 failure occurs, the hazard rate is updated, and two possibilities can occur:

   (a) If the updated hazard rate is above the opportunistic maintenance level (W), then the transition probability is given by:

$$P_{(z,(n-1),(r-1)),(PM1)} = P(PM1, \xi'_2 < \xi'_1, \xi_2 < r\Delta \mid n\Delta < T_1, r\Delta < T_2, \xi_1 > (n-1)\Delta, \xi_2 > (r-1)\Delta, Z_{(n-1)\Delta} = z) = \int_0^{\Delta} \sum_{z'} P_{z,z'}(t) \cdot \overline{R}_1((n-1), z, t) \cdot f_2(t|(r-1))\mathrm{d}t, \tag{15}$$

   where $\xi'_1 = \xi_1 - (n-1)\Delta, \xi'_2 = \xi_2 - (r-1)\Delta$, z' are the covariate values that cause the hazard rate to exceed the opportunistic maintenance level and

$$f_2(t \mid (r-1)) = \frac{\mathrm{d}}{\mathrm{d}t} P(\xi_2 \leq ((r-1)\Delta + t) \mid \xi_2 > (r-1)\Delta). \tag{16}$$

   (b) When the updated hazard rate is below the opportunistic maintenance level (W), the transition probability is obtained as follows:

$$P_{(z,(n-1),(r-1)),(z',n,1)} = P(Z_{n\Delta} = z', \xi'_2 < \xi'_1, \xi_2 < r\Delta \mid n\Delta < T_1, r\Delta < T_2, \xi_1 > (n-1)\Delta, \xi_2 > (r-1)\Delta, Z_{(n-1)\Delta} = z) = \int_0^{\Delta} P_{z,z'}(t) \cdot \overline{R}_1((n-1), z, t) \cdot f_2(t|(r-1))\mathrm{d}t, \tag{17}$$

   where z' is the covariate value that causes the hazard rate not to exceed the opportunistic maintenance level.

5. When unit 1 is in the $PM1$ or $PM2$ state, then mandatory replacement of unit 1 is performed and the system goes back to state (0, 0, 0). We have:

$$P_{(PM1),(0,0,0)} = P_{(PM2),(0,0,0)} = 1. \tag{18}$$

6. When unit 1 failure happens, then the next state will be (0, 0, 0) and the transition probability $P_{(z,(n-1),(r-1)),(0,0,0)}$ from state $(z, (n-1), (r-1))$ to state (0, 0, 0), where $h((n-1)\Delta, z) < U$, can be calculated as follows:

$$P_{(z,(n-1),(r-1)),(0,0,0)} = 1 - \overline{R}_1((n-1), z, t). \tag{19}$$

In the next two sections, the formulas for the calculation of the expected sojourn times and expected cost are developed.

3.2. Expected sojourn times

The expected sojourn time incurred until the next decision epoch for the state (z, n, r) where $h(n\Delta, z) < U$ can be obtained by using Theorem 1.

Theorem 1 *The expected sojourn time given the state is* (z, n, r) *where* $h(n\Delta, z) < U$ *is given by:*

$$\begin{aligned}\tau_{(z,n,r)} &= \Delta \cdot \sum_{z'} P_{(z,n,r),(z',(n+1),(r+1))} + \Delta \\ &\cdot \sum_{z'} P_{(z,n,r),(z',(n+1),1)} + \int_0^{\Delta} t \cdot \sum_{z'} P_{z,z'}(t) \cdot \overline{R}_1(n,z,t) \\ &\cdot f_2(t|r)\mathrm{d}t + \Delta\Big(\sum_{z'} P_{z,z'}(\Delta)\cdot \overline{R}_1(n,z,\Delta)\cdot \frac{R_2((r+1)\Delta)}{R_2(r\Delta)} \\ &\cdot I_{(n+1)\Delta<T_1} + \overline{R}_1(n,z,\Delta)\cdot \frac{R_2((r+1)\Delta)}{R_2(r\Delta)}\cdot I_{(n+1)\Delta \geq T_1}\Big) \\ &+ \Delta \cdot \sum_{z'} P_{z,z'}(\Delta)\cdot \overline{R}_1(n,z,\Delta)\cdot \frac{R_2((r+1)\Delta)}{R_2(r\Delta)} + \Delta \\ &\cdot P_{(z,n,r)(z',(n+1),1)} + \int_0^{\Delta} (t+T_F)\cdot\big(1-\overline{R}_1(n,z,t)\big)\mathrm{d}t.\end{aligned}$$

Proof See "Appendix 2." □

If the age of unit 1 exceeds the predetermined age (T_1) or hazard rate crosses the preventive maintenance level, then the sojourn time in *PM*2 state will be:

$$\tau_{PM2} = T_P, \tag{20}$$

whereas upon unit 2 failure or its preventive maintenance time (T_2), if unit 1 updated hazard rate is between the opportunistic and preventive maintenance level, then the opportunistic maintenance will be performed and the sojourn time in *PM*1 state will be as follows:

$$\tau_{PM1} = T_P. \tag{21}$$

3.3. Expected cost

The average cost incurred until the next decision epoch for state (z, n, r) where it does not cross the preventive maintenance level, can be obtained by using Theorem 2.

Theorem 2 *The average cost given the state is* (z, n, r) *where* $h(n\Delta, z) < U$ *is given by:*

$$\begin{aligned}C_{(z,n,r)} &= C_I \cdot \sum_{z'} P_{(z,n,r)(z',(n+1),(r+1))} + (C_s + C_{F2} + C_{OP}) \\ &\cdot \sum_{z'} P_{(z,n,r)(z',(n+1),1)} + (C_s + C_{F2} + C_{OP})\cdot \int_0^{\Delta} \sum_{z'} P_{z,z'}(t) \\ &\cdot \overline{R}_1(n,z,t)\cdot f_2(t\,|\,r)\mathrm{d}t + (C_s + C_{P1} + C_{OP})\cdot\Big(\sum_{z'} P_{z,z'}(\Delta) \\ &\cdot \overline{R}_1(n,z,\Delta)\cdot \frac{R_2((r+1)\Delta)}{R_2(r\Delta)} + \overline{R}_1(n,z,\Delta)\cdot \frac{R_2((r+1)\Delta)}{R_2(r\Delta)}\Big) \\ &+ (C_s + C_{P2} + C_{OP})\cdot\big(\sum_{z'} P_{z,z'}(\Delta)\cdot \overline{R}_1(n,z,\Delta) \\ &\cdot \frac{R_2((r+1)\Delta)}{R_2(r\Delta)} + (C_s + C_{P2} + C_{OP})\cdot P_{(z,n,r)(z',(n+1),1)} \\ &+ (C_s + C_{F1} + C_{P2} + C_{OP} + C_{LP}T_F)\cdot \int_0^{\Delta} \big(1-\overline{R}_1(n,z,t)\big)\mathrm{d}t.\end{aligned} \tag{22}$$

Proof See "Appendix 3." □

The average cost incurred in *PM*2 state is as follows:

$$C_{PM2} = C_{LP}.T_P + C_{P2}. \tag{23}$$

Upon unit 1 opportunistic maintenance, the system will be in *PM*1 state with the expected cost:

$$C_{PM1} = C_{LP}.T_P + C_{P1}. \tag{24}$$

Now, all the SMDP quantities have been determined. In order to find the optimal policy in the SMDP framework, different approaches have been introduced. One of the approaches which is widely used in various applications is the policy iteration algorithm which has been applied widely [e.g., in health care (Schaefer *et al*, 2004), queuing systems (Xia *et al*, 2009), and airline industry (Gosavi, 2004)]. The policy iteration algorithm is an efficient algorithm that enables to obtain the optimal policy very fast (see Tijm, 1994, p. 171). We will apply this algorithm by choosing the initial policy to find the corresponding relative values and the average cost and then iteratively repeat the algorithm until the optimal policy is found.

Before proceeding to experimental section where we use real data, we first derive a formula for the mean residual life which is an important statistic in practical applications.

4. Residual life prediction

In this section, we derive the explicit formula for the system mean residual time to failure.

Lemma 1 *For any state* (z, n, r)*, the mean residual life* $MRL_{(z,n,r)}$ *is given by:*

$$MRL_{(z,n,r)} = \int_0^{\infty} \overline{R}_1(n,z,t)\cdot \frac{R_2(r\Delta + t)}{R_2(r\Delta)}\mathrm{d}t. \tag{25}$$

Proof Suppose $\xi_2' = \xi_2 - r\Delta$ and $\xi_1' = \xi_1 - n\Delta$, then for any $t \in R^+$, the mean residual life is given by:

$$E\left\{\xi \mid \xi_1 > n\Delta, \xi_2 > r\Delta, z\right\}$$
$$= \int_0^\infty P\left(\xi_1' > t, \xi_2' > t \mid \xi_1' > \Delta, \xi_2' > \Delta, z\right)\mathrm{d}t$$
$$= \int_0^\infty \overline{R}_1(n,z,t)\cdot\frac{R_2(r\Delta+t)}{R_2(r\Delta)}\mathrm{d}t.$$

where $\xi = min(\xi_1, \xi_2)$ and term $E(\cdot)$ denotes expectation operator. The conditional reliability function of unit 1 is provided by Eq. (9). □

5. Experimental results

We study real diagnostic data of the heavy hauler trucks in a mining company considering transmission and clutch as the two main units. Transmission caused frequent unpredicted failures and was therefore subject to CM, and the age information of the second main unit (clutch) is available. The remaining units such as bearings, engine belt, radial shaft seal, and rings are considered cheaper and easily adjustable or replaceable units when the opportunity occurs. During the operational life of the transmissions, oil data measurements were collected. The total number of recorded histories was 51 which consists of 20 failure and 31 suspension histories. The application of the EXAKT software (Banjevic and Jardine, 2006) suggested that iron was the most significant covariate in the PH model. The baseline hazard rate is Weibull, $h_0(t) = \frac{\beta_1 t^{\beta_1 - 1}}{\alpha_1^{\beta_1}}$, where $\alpha_1 = 21632.3$ and $\beta_1 = 1.78563$ and $\psi(Z_t) = \exp(0.0468681\mathrm{z(t)})$.

The iron covariate values are determined through CM at equidistant inspection epochs, and the deterioration process is described by a continuous-time Markov process with state space $\Omega = \{0, 1, 2\}$. The coded states 0 and 1 represent the healthy and warning operational states, respectively, and state 2 corresponds to the degraded absorbing state. The following ranges for iron values were considered: {0–20, 20–70, 70 and over} where the coded values 0, 1, and 2 represent the values 10, 45, and 85, respectively. The transition rate matrix of the iron covariate is obtained based on the modified states and the analysis which was performed in Banjevic and Jardine (2006), Makis *et al* (2006) and Kim *et al* (2011) as follows:

$$Q \times 10^4 = \begin{bmatrix} -3.506 & 3.586 & 0.004 \\ 0 & -6.414 & 6.414 \\ 0 & 0 & 0 \end{bmatrix}.$$

The age information of clutch is available, and its lifetime distribution is Weibull with parameters $\alpha_2 = 18730$ and $\beta_2 = 2.88$. The system preventive and replacement time parameters are given by: $T_P = 1$ and $T_F = 1$ hour, where T_P is the time to perform preventive maintenance and T_F is the time to recover the system upon transmission failure.

Table 1 Optimal maintenance policy for a multi-unit series system using PHM

Decision variables	*Optimal value*
Opportunistic maintenance level (W^*)	8.5315 $\times 10^{-4}$
Preventive maintenance level (U^*)	0.0013
Inspection interval (Δ^*)	590
Preventive maintenance time of unit 2 (T_2^*)	8260
Optimal average cost (g^*)	0.3797

The following costs will be considered in the experiment: $C_I = 10, C_{LP} = 100, C_{F1} = 6780,$ $C_{P1} = 1560, C_{P2} = 500, C_{F2} = 1200, C_{OP} = 100, C_s = 350.$ We have computed the optimal inspection interval and the opportunistic and preventive maintenance levels minimizing the long-run expected average cost per unit time. The maximum value of the hazard rate $H = 6.4 \times 10^{-3}$ is derived based on $z = 2$ and the age of a transmission equal to 34500 hours when it is working in the healthy state ($z = 0$) and its reliability function at this age is 0.1. Then, the discretization level 30 is chosen ($L = 30$). The results are obtained using Eq. (7), and the policy iteration algorithm is shown in Table 1.

An example of the hazard rate plot with the opportunistic and preventive maintenance levels is shown in Figure 1 for one of the failure histories.

The green stars show the values of the hazard rate at each inspection epoch. Clutch failure occurs before the eighth inspection epoch, and then, the hazard rate is updated at this epoch, as shown by red circle in Figure 1. The updated hazard rate exceeds the opportunistic maintenance level (W), so all units are replaced opportunistically.

Figure 2 shows the value of the reliability function at each inspection epoch, which is decreasing and it goes to zero upon clutch failure. Also, the reliability function decreases considerably from inspection epoch 5 to 6, because the oil analysis revealed that the covariate (*iron*) is in the warning state. Finally, Figure 3 illustrates the MRL of the system. As it is shown, when the updated hazard rate of the transmission exceeds the opportunistic maintenance level, the MRL value is low and it is the time to opportunistically maintain both units in the system.

Since the exact values of the cost components may be difficult to determine in practical applications, sometimes the estimates are provided. Therefore, to investigate the performance of the proposed model, using different cost parameters, a designed experiment is performed. The long-run expected average cost per unit time is the response variable to identify which cost parameters and their interactions are significant using a 2^k factorial design. We have selected five cost parameters which include inspection cost C_I, lost sales cost C_{LP}, failure cost ratio of transmission $\frac{C_{F1}}{C_{P1}}$, failure cost ratio of clutch $\frac{C_{F2}}{C_{P2}}$, and set-up cost C_s. We included the adjustment cost of $(N-2)$ units in the set-up cost parameter to decrease the

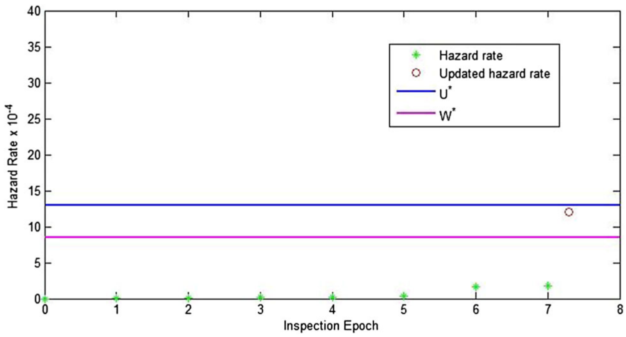

Figure 1 Graphical representation of the hazard rate evolution using the proposed maintenance model.

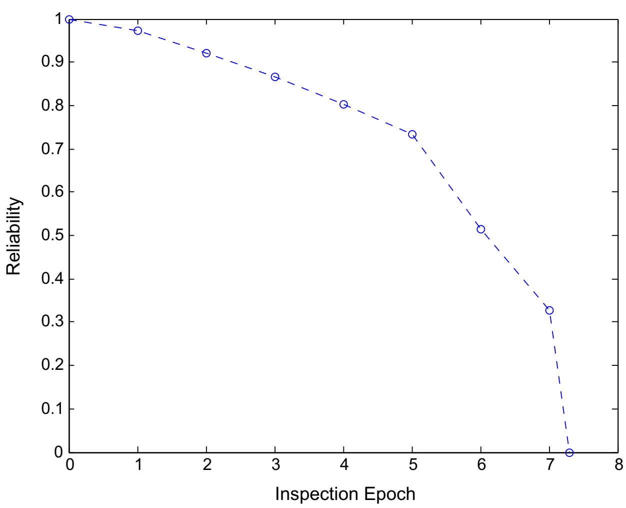

Figure 2 Plot of the reliability function at each inspection epoch.

number of factors and performed a full factorial designed experiment. We choose two levels for each factor which are summarized in Table 2.

We obtained the response variable values considering different combinations of the factor levels using the proposed SMDP algorithm. The designed experiment analysis reveals that none of the third- and higher-order interactions is significant. Figure 4 shows the normal probability plot of all the factors and their interactions.

The plot indicates that the cost ratio of the first unit (transmission), set-up cost, and interaction between the first unit failure cost ratio with the second unit (clutch) cost ratio, and the lost sales cost are identified as significant factors and the rest are nonsignificant ones. However, the second-order interactions (*CD* and *CE*) are very close to the noise line in the normal plot, and clearly, the first unit cost ratio and the set-up cost are more significant. Thus, it is interesting to further investigate the effect of these factors on the decision variables by performing sensitivity analysis.

Table 3 shows the optimal policies and the average costs for varying set-up cost. When the set-up cost increases, the

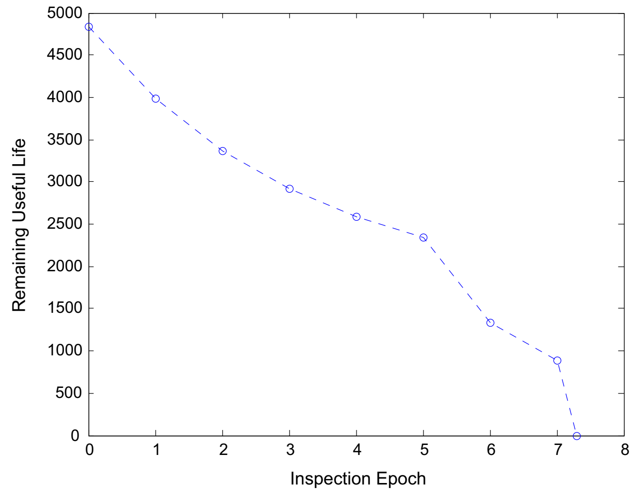

Figure 3 Mean residual life (MRL) of the proposed maintenance model.

Table 2 Factors used in the designed experiment

	Design factors				
Level	*A* C_I	*B* C_{LP}	*C* $\frac{C_{F1}}{C_{P1}}$	*D* $\frac{C_{F2}}{C_{P2}}$	*E* C_s
Low (−)	2	20	3	1.5	200
High (+)	20	200	6	4	1000

opportunistic maintenance occurs more frequently to jointly maintain both units. The results indicate that the inspection frequency increases, and the opportunistic maintenance level decreases. It is interesting to observe that for the values of the set-up cost greater than 600, the opportunistic level remains constant, but the inspection interval decreases because the stopping of the system becomes more costly.

Another significant factor is the failure cost ratio of the first main unit (transmission). The results in Table 4 indicate that when this ratio increases, inspection is performed more frequently and the preventive maintenance level decreases as well to reduce costly failures, because the higher ratio means that the failure cost is considerably higher than the preventive maintenance cost.

5.1. Comparison with other policies

In this section, we compare the performance of our proposed maintenance policy with other policies, (1) considering just a preventive maintenance policy without opportunistic maintenance and (2) the corrective maintenance policy.

First, we investigate the effect of the opportunistic maintenance level on the optimal maintenance cost for the proposed system.

For the policy without opportunistic maintenance level, there is no opportunity to perform PM on transmission upon

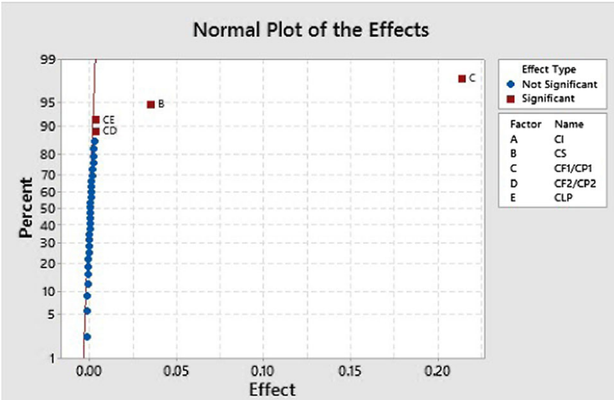

Figure 4 Normal probability plot of the effect estimates for all factors and their interactions.

Table 3 Results of the sensitivity analysis considering varying set-up cost

C_S	*200*	*400*	*600*	*800*	*1000*
$W^* \times 10^{-4}$	8.5315	8.5315	6.3986	6.3986	6.3986
$U^* \times 10^{-4}$	13	13	13	11	11
Δ^*	620	590	560	520	500
T_2^*	8060	8260	7840	8320	8500
g^*	0.3670	0.3714	0.3846	0.4012	0.4253

Table 4 Results of the sensitivity analysis considering varying first unit failure cost ratio

$\frac{C_F1}{C_P1}$	*3*	*4*	*5*	*6*
$W^* \times 10^{-4}$	8.5315	8.5315	8.5315	6.3986
$U^* \times 10^{-4}$	15	13	11	11
Δ^*	550	560	530	480
T_2^*	7700	7840	7950	8160
g^*	0.2741	0.3431	0.4189	0.5086

clutch failure. As shown in Table 5, the minimum long-run expected average cost is equal to 0.4602 so the results show that the policy with the opportunistic maintenance level is more economical than the optimal maintenance policy without the opportunistic maintenance level.

Next, we compare the proposed approach with the well-known corrective maintenance policy which does not take the CM information into account. In this case, the maintenance activities are performed on each unit separately, without considering economic dependency between the units. Since the actions are performed independently, the set-up cost is incurred each time when a maintenance action is performed and also the lack of the condition monitoring information causes that the deterioration level of transmission is not taken into account when making maintenance decisions. The optimal average cost for the corrective maintenance policy is 0.6182, which is a significant increase (38.57%) and again confirms the superiority of the proposed maintenance policy using PHM.

6. Conclusions and future research

In this paper, we have developed a model and a computational algorithm that can be used to determine the optimal maintenance policy for a multi-unit series system where one unit is subject to condition monitoring, while just the age information is available for unit 2, which has a general distribution. The other units are adjusted or replaced each time the system is maintained. Unit 1 deterioration is described by a PH model, where the covariate evolution is modeled as a continuous-time Markov process. We have developed a computational algorithm in the SMDP framework to minimize the long-run expected average cost per unit time for the whole system. A real application of the proposed model using oil data from a mining company has been provided. Also, the comparison of the opportunistic maintenance policy using PHM with other policies (with no opportunistic maintenance level and the corrective maintenance policy) confirms the superiority of the proposed model.

We also suggest a few possible directions for future research. First, a more general model can be developed by considering a larger number of states for unit 1 covariate process and increasing the number of units to which preventive maintenance is applied. Another extension could be to consider more general distributions for the sojourn times of the covariate process in the operational states, such as Erlang or Weibull distribution. We also assumed that the preventive maintenance actions bring the system back to the “as good as new” condition, while there are some papers considering different assumptions, e.g., Wang and Christer (2000) where a preventive action as a maintenance activity may restore the system to a better or possibly worse condition depending on the quality and nature of the action, or Makis and Jardine (1992) considered a model incorporating imperfect repair to find the optimal replacement policy. Although the cost models are more common in the maintenance literature, availability

Table 5 Comparison with other policies

	Proposed model	*No opportunistic maintenance*	*Corrective maintenance policy*
Preventive maintenance level (U^*)	0.0013	0.0015	–
Optimal inspection interval (Δ^*)	590	550	–
Optimal average cost (g^*)	0.3797	0.4602	0.6182

models are preferable in some situations where it is difficult to estimate the cost parameters. (e.g., Jiang *et al*, 2013), which can be another direction for future research.

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Appendix 1: Reliability function of unit 1

Let assume that the sojourn time of the covariate process Z in the healthy and unhealthy states $i = 0, 1$ is exponentially distributed with parameters v_i.

$$\begin{aligned}\overline{R_1}(n,z,t) &= P(\xi > n\Delta + t \mid \xi > n\Delta, Z_{1\Delta}, \ldots, Z_{n\Delta}, Z_{n\Delta} = z)\\ &= E\left[\exp\left(-\int_{n\Delta}^{n\Delta+t} h_0(s)\psi(Z_s)ds\right) \mid Z_{n\Delta} = z\right]\end{aligned} \tag{26}$$

If $z = 0$, then Eq. (26) can be written as follows:

$$\begin{aligned}\overline{R_1}(n,0,t) = &\int_0^\infty E\left[\exp\left(-\int_{n\Delta}^{n\Delta+t} h_0(s)\psi(Z_s)ds\right) \mid t_0 = u\right]\\ &\cdot p_{01}\cdot v_0 e^{-v_0 u}du\\ &+\int_0^\infty E\left[\exp\left(-\int_{n\Delta}^{n\Delta+t} h_0(s)\psi(Z_s)ds\right) \mid t_0 = u\right]\\ &\cdot p_{02}\cdot v_0 e^{-v_0 u}du\end{aligned} \tag{27}$$

where t_0 is the time that the covariate process is in the healthy state. The first part of the above equation can be extended by splitting the integral, as follows:

$$\begin{aligned}&\int_t^\infty E\left[\exp\left(-\int_{n\Delta}^{n\Delta+t} h_0(s)\psi(Z_s)ds\right)|t_0=u\right]\cdot p_{01}\cdot v_0 e^{-v_0 u}du\\ &\quad+\int_0^t E\left[\exp\left(-\int_{n\Delta}^{n\Delta+t} h_0(s)\psi(Z_s)ds\right)|t_0=u\right]\cdot p_{01}\cdot v_0 e^{-v_0 u}du\\ &=\int_t^\infty \exp\left(-\int_{n\Delta}^{n\Delta+t} h_0(s)\psi(0)ds\right)\cdot p_{01}\cdot v_0 e^{-v_0 u}du\\ &\quad+\int_0^t E\left[\exp\left(-\int_{n\Delta}^{n\Delta+t} h_0(s)\psi(Z_s)ds\right)|t_0=u, Z_{t_0}=1\right]\\ &\quad\cdot p_{01}\cdot v_0 e^{-v_0 u}du\end{aligned} \tag{28}$$

The second part of Eq. (28) can be determined by conditioning on the sojourn time in the warning state, so we have:

$$\begin{aligned}&\int_0^t\int_u^\infty E\left[\exp\left(-\int_{n\Delta}^{n\Delta+t} h_0(s)\psi(Z_s)ds\right) \mid t_0=u, Z_{t_0}=1,\right.\\ &\quad\left. t_1 = v\right]\cdot p_{01}\cdot v_0 e^{-v_0 u}\cdot v_1 e^{-v_1 v}dvdu\\ &=\int_0^t\int_u^t \exp\left(-\left(\int_{n\Delta}^{n\Delta+u} h_0(s)\psi(0)ds+\int_{n\Delta+u}^{n\Delta+v} h_0(s)\psi(1)ds\right.\right.\\ &\quad\left.\left.+\int_{n\Delta+v}^{n\Delta+t} h_0(s)\psi(2)ds\right)\right)\cdot p_{01}\cdot v_0 e^{-v_0 u}\cdot v_1 e^{-v_1 v}dvdu\\ &\quad+\int_0^t\int_t^\infty \exp\left(-\left(\int_{n\Delta}^{n\Delta+u} h_0(s)\psi(0)ds+\int_{n\Delta+u}^{n\Delta+t} h_0(s)\psi(1)ds\right)\right)\\ &\quad\cdot p_{01}\cdot v_0 e^{-v_0 u}\cdot v_1 e^{-v_1 v}dvdu\end{aligned}$$

The second part of Eq. (27) can be written by splitting the integral as follows:

$$\begin{aligned}&\int_t^\infty E\left[\exp\left(-\int_{n\Delta}^{n\Delta+t} h_0(s)\psi(Z_s)ds\right)|t_0=u\right]\cdot p_{02}\cdot v_0 e^{-v_0 u}du\\ &\quad+\int_0^t E\left[\exp\left(-\int_{n\Delta}^{n\Delta+t} h_0(s)\psi(Z_s)ds\right)|t_0=u\right]\cdot p_{02}\cdot v_0 e^{-v_0 u}du\\ &=\int_t^\infty \exp\left(-\int_{n\Delta}^{n\Delta+t} h_0(s)\psi(0)ds\right)\cdot p_{02}\cdot v_0 e^{-v_0 u}du\\ &\quad+\int_0^t \exp\left(-\left(\int_{n\Delta}^{n\Delta+u} h_0(s)\psi(0)ds+\int_{n\Delta+u}^{n\Delta+t} h_0(s)\psi(2)ds\right)\right)\\ &\quad\cdot p_{02}\cdot v_0 e^{-v_0 u}du\end{aligned}$$

So, the reliability function in Eq. (27) is given by:

$$\begin{aligned}\overline{R_1}(n,0,t) = &\exp\left(-\int_{n\Delta}^{n\Delta+t} h_0(s)\psi(0)ds\right)e^{-v_0 t}\\ &+\int_0^t\int_u^t \exp\left(-\left(\int_{n\Delta}^{n\Delta+u} h_0(s)\psi(0)ds+\int_{n\Delta+u}^{n\Delta+v} h_0(s)\psi(1)ds\right.\right.\\ &\left.\left.+\int_{n\Delta+v}^{n\Delta+t} h_0(s)\psi(2)ds\right)\right)\cdot p_{01}\cdot v_0 e^{-v_0 u}\cdot v_1 e^{-v_1 v}dvdu\\ &+\int_0^t \exp\left(-\left(\int_{n\Delta}^{n\Delta+u} h_0(s)\psi(0)ds+\int_{n\Delta+u}^{n\Delta+t} h_0(s)\psi(1)ds\right)\right)\\ &\cdot p_{01}\cdot v_0 e^{-v_0 u}e^{-v_1 t}du \int_0^t \exp\left(-\left(\int_{n\Delta}^{n\Delta+u} h_0(s)\psi(0)ds\right.\right.\\ &\left.\left.+\int_{n\Delta+u}^{n\Delta+t} h_0(s)\psi(2)ds\right)\right)\cdot p_{02}\cdot v_0 e^{-v_0 u}du\end{aligned}$$

where $p_{01} = \frac{q_{01}}{q_{01}+q_{02}}$, and $p_{02} = \frac{q_{02}}{q_{01}+q_{02}}$.

When $z = 1$, then Eq. (26) is given by:

$$\begin{aligned}\overline{R_1}(n,1,t) &= \int_0^\infty E\left[\exp\left(-\int_{n\Delta}^{n\Delta+t} h_0(s)\psi(Z_s)ds\right)|t_1 = v\right]\cdot v_1\\ e^{-v_1 v}dv &= \int_t^\infty \exp\left(-\int_{n\Delta}^{n\Delta+t} h_0(s)\psi(1)ds\right)\cdot v_1\cdot e^{-v_1 v}dv\\ &\quad+\int_0^t E\left[\exp\left(-\int_{n\Delta}^{n\Delta+t} h_0(s)\psi(Z_s)ds\right)|t_1\right.\\ &\quad\left.= v, Z_{t_1} = 2\right]v_1\cdot e^{-v_1 v}dv\\ &= \exp\left(-\int_{n\Delta}^{n\Delta+t} h_0(s)\psi(1)ds\right)\cdot e^{-v_1 t}\\ &\quad+\int_0^t \exp\left(-\left(\int_{n\Delta}^{n\Delta+v} h_0(s)\psi(1)ds\right.\right.\\ &\quad\left.\left.+\int_{n\Delta+v}^{n\Delta+t} h_0(s)\psi(2)ds\right)\right)\\ &\quad v_1\cdot e^{-v_1 v}dv.\end{aligned}$$

Finally, the conditional reliability when $z=2$ is as follows:

$$\overline{R_1}(n,2,t)=E\Big[\exp\Big(-\int_{\mathrm{n}\Delta}^{\mathrm{n}\Delta+\mathrm{t}}\mathrm{h}_0(\mathrm{s})\psi(\mathrm{Z_s})\mathrm{ds}\Big)\mid \mathrm{Z_s}=2\Big]$$
$$=\exp\Big(-\int_{\mathrm{n}\Delta}^{\mathrm{n}\Delta+\mathrm{t}}\mathrm{h}_0(\mathrm{s})\psi(2)\mathrm{ds}\Big).$$

Appendix 2: Proof of Theorem 1

Suppose $\xi_1'=\xi_1-n\Delta$, and $\xi_2'=\xi_2-\mathrm{r}\Delta$, the expected sojourn time given the state is (z, n, r) where $h((n-1)\Delta,z)<U$ is given by:

$$\begin{aligned}
\tau_{(z,n,r)}=&E(\text{Sojourn time}|\xi_1>n\Delta,\xi_2>r\Delta,n\Delta<T_1,\\
&r\Delta<T_2,h(n\Delta,z)<U)=\sum_{z'}E(\text{Sojourn time}|\xi_1>n\Delta,\\
&\xi_2>r\Delta,n\Delta<T_1,r\Delta<T_2,h(n\Delta,z)<U,\xi_1>(n+1)\Delta,\\
&\xi_2>(r+1)\Delta,(n+1)\Delta<T_1,(r+1)\Delta<T_2,\\
&h((n+1)\Delta,z')<U)\cdot P_{(z,n,r)(z',(n+1),(r+1))}+\sum_{z'}E(\text{Sojourn}\\
&\text{time}|\xi_1>\mathrm{n}\Delta,\xi_2>\mathrm{r}\Delta,\mathrm{h(n\Delta,z)<U,n\Delta<T_1,r\Delta<T_2},\\
&\xi_2'<\xi_1',\xi_2<(r+1)\Delta,(n+1)\Delta<T_1,(r+1)\Delta<T_2,\\
&h(n\Delta+\xi_2',z')<W)\cdot P_{(z,n,r)(z',(n+1),1)}+E(\text{Sojourn time}|\\
&\xi_1>n\Delta,\xi_2>r\Delta,n\Delta<T_1,r\Delta<T_2,h(n\Delta,z)<U,\xi_2'<\xi_1',\\
&\xi_2<(r+1)\Delta,(n+1)\Delta<T_1,(r+1)\Delta<T_2,h(n\Delta+\xi_2',z')\geq W)\\
&\cdot P_{(z,n,r)(PM1)}+E(\text{Sojourn time}|\xi_1>n\Delta,\xi_2>r\Delta,\\
&h(n\Delta,z)<U,\xi_1>(n+1)\Delta,\xi_2>(r+1)\Delta,(n+1)\Delta<T_1,\\
&(r+1)\Delta<T_2,h((n+1)\Delta,z')\geq U)\cdot P_{(z,n,r)(PM2)}+E(\text{Sojourn}\\
&\text{time}|\xi_1>\mathrm{n}\Delta,\xi_2>\mathrm{r}\Delta,\mathrm{n\Delta<T_1,r\Delta<T_2,h(n\Delta,z)<U},\\
&\xi_1>(n+1)\Delta,\xi_2>(r+1)\Delta,(n+1)\Delta\geq T_1,(r+1)\Delta<T_2)\\
&\cdot P_{(z,n,r)(PM2)}+E(\text{Sojourn time}|\xi_1>n\Delta,\xi_2>r\Delta,n\Delta<T_1,\\
&r\Delta<T_2,\xi_1>(n+1)\Delta,\xi_2>(r+1)\Delta,(n+1)\Delta<T_1,\\
&(r+1)\Delta\geq T_2,h((n+1)\Delta,z)\geq W)\cdot P_{(z,n,r)(PM1)}+\\
&E(\text{Sojourn time}|\xi_1>n\Delta,\xi_2>r\Delta,n\Delta<T_1,r\Delta<T_2,\\
&\xi_1>(n+1)\Delta,\xi_2>(r+1)\Delta,(n+1)\Delta<T_1,(r+1)\Delta\geq T_2,\\
&h((n+1)\Delta,z)<W)\cdot P_{(z,n,r)(z',(n+1),1)}+E(\text{Sojourn}\\
&\text{time}|\xi_1>\mathrm{n}\Delta,\xi_2>\mathrm{r}\Delta,\mathrm{n\Delta<T_1,r\Delta<T_2,h(n\Delta,z)<U},\\
&\xi_1'<\xi_2',\xi_1<(n+1)\Delta,(n+1)\Delta<T_1)\cdot P_{(z,n,r)(0,0,0)}\\
=&\Delta\cdot\sum_{z'}P_{(z,n,r)(z',(n+1),(r+1))}+\Delta\cdot\sum_{z'}P_{(z,n,r)(z',(n+1),1)}\\
&+\int_0^\Delta t\cdot\sum_{z'}P_{z,z'}(t)\cdot\overline{R}_1(n,z,t)\cdot f_2(t|r)\mathrm{d}t\\
&+\Delta\cdot\Big(\sum_{z'}P_{z,z'}(\Delta)\cdot\overline{R}_1(n,z,\Delta)\cdot\frac{R_2((r+1)\Delta)}{R_2(r\Delta)}\cdot I_{(n+1)\Delta<T_1}\\
&+\overline{R}_1(n,z,\Delta)\cdot\frac{R_2((r+1)\Delta)}{R_2(r\Delta)}\cdot I_{(n+1)\Delta\geq T_1}\Big)+\Delta\cdot\sum_{z'}P_{z,z'}(\Delta)\\
&\cdot\overline{R}_1(n,z,\Delta)\cdot\frac{R_2((r+1)\Delta)}{R_2(r\Delta)}+\Delta\cdot\sum_{z'}P_{(z,n,r)(z',(n+1),1)}\\
&+\int_0^\Delta(t+T_F)\cdot\big(1-\overline{R}_1(n,z,t)\big)\mathrm{d}t.
\end{aligned}$$

Appendix 3: Proof of Theorem 2

Suppose $\xi_1'=\xi_1-n\Delta$, and $\xi_2'=\xi_2-\mathrm{r}\Delta$, then the expected cost given the state is (z, n, r) where $h((n-1)\Delta,z)<U$ is given by:

$$\begin{aligned}
C_{(z,n,r)}=&E(\mathrm{Cost}|\xi_1>\mathrm{n}\Delta,\xi_2>\mathrm{r}\Delta,\mathrm{n\Delta<T_1,r\Delta<T_2},\\
&h(n\Delta,z)<U)=\sum_{z'}E(\mathrm{Cost}|\xi_1>\mathrm{n}\Delta,\xi_2>\mathrm{r}\Delta,\mathrm{n\Delta<T_1},\\
&r\Delta<T_2,h(n\Delta,z)<U,\xi_1>(n+1)\Delta,\xi_2>(r+1)\Delta,\\
&(n+1)\Delta<T_1,(r+1)\Delta<T_2,h((n+1)\Delta,z')<U)\\
&\cdot P_{(z,n,r)(z',(n+1),(r+1))}+\sum_{z'}E(\mathrm{Cost}|\xi_1>\mathrm{n}\Delta,\xi_2>\mathrm{r}\Delta,\\
&h(n\Delta,z)<U,n\Delta<T_1,r\Delta<T_2,\xi_2'<\xi_1',\xi_2<(r+1)\Delta,\\
&(n+1)\Delta<T_1,(r+1)\Delta<T_2,h(n\Delta+\xi_2',z')<W)\\
&\cdot P_{(z,n,r)(z',(n+1),1)}+E(\mathrm{Cost}|\xi_1>\mathrm{n}\Delta,\xi_2>\mathrm{r}\Delta,\mathrm{n\Delta<T_1},\\
&r\Delta<T_2,h(n\Delta,z)<U,\xi_2'<\xi_1',\xi_2<(r+1)\Delta,(n+1)\Delta<T_1,\\
&(r+1)\Delta<T_2,h(n\Delta+\xi_2',z')\geq W)\cdot P_{(z,n,r)(PM1)}+\\
&E(\mathrm{Cost}|\xi_1>\mathrm{n}\Delta,\xi_2>\mathrm{r}\Delta,\mathrm{h(n\Delta,z)<U},\xi_1>(\mathrm{n}+1)\Delta,\\
&\xi_2>(r+1)\Delta,(n+1)\Delta<T_1,(r+1)\Delta<T_2,h((n+1)\Delta,z')\geq U)\\
&\cdot P_{(z,n,r)(PM2)}+E(\mathrm{Cost}|\xi_1>\mathrm{n}\Delta,\xi_2>\mathrm{r}\Delta,\mathrm{n\Delta<T_1,r\Delta<T_2},\\
&h(n\Delta,z)<U,\xi_1>(n+1)\Delta,\xi_2>(r+1)\Delta,(n+1)\Delta\geq T_1,\\
&(r+1)\Delta<T_2)\cdot P_{(z,n,r)(PM2)}+E(\mathrm{Cost}|\xi_1>\mathrm{n}\Delta,\xi_2>\mathrm{r}\Delta,\\
&n\Delta<T_1,r\Delta<T_2,\xi_1>(n+1)\Delta,\xi_2>(r+1)\Delta,\\
&(n+1)\Delta<T_1,(r+1)\Delta\geq T_2,h((n+1)\Delta,z)\geq W)\\
&\cdot P_{(z,n,r)(PM1)}+E(\mathrm{Cost}|\xi_1>\mathrm{n}\Delta,\xi_2>\mathrm{r}\Delta,\mathrm{n\Delta<T_1,r\Delta<T_2},\\
&\xi_1>(n+1)\Delta,\xi_2>(r+1)\Delta,(n+1)\Delta<T_1,(r+1)\Delta\geq T_2,\\
&h((n+1)\Delta,z)<W)\cdot P_{(z,n,r)(z',(n+1),1)}+E(\mathrm{Cost}|\\
&\xi_1>n\Delta,\xi_2>r\Delta,n\Delta<T_1,r\Delta<T_2,h(n\Delta,z)<U,\\
&\xi_1'<\xi_2',\xi_1<(n+1)\Delta,(n+1)\Delta<T_1)\cdot P_{(z,n,r)(0,0,0)}\\
=&C_I\cdot\sum_{z'}P_{(z,n,r)(z',(n+1),(r+1))}+(C_s+C_{F2}+C_{OP})\\
&\cdot\sum_{z'}P_{(z,n,r)(z',(n+1),1)}+(C_s+C_{F2}+C_{OP})\cdot\int_0^\Delta\sum_{z'}P_{z,z'}(t)\\
&\cdot\overline{R}_1(n,z,t)\cdot f_2(t\,|\,r)\mathrm{d}t+(C_s+C_{P1}+C_{OP})\cdot\Big(\sum_{z'}P_{z,z'}(\Delta)\\
&\cdot\overline{R}_1(n,z,\Delta)\cdot\frac{R_2((r+1)\Delta)}{R_2(r\Delta)}+\overline{R}_1(n,z,\Delta)\cdot\frac{R_2((r+1)\Delta)}{R_2(r\Delta)}\Big)\\
&+(C_s+C_{P2}+C_{OP})\cdot(\sum_{z'}P_{z,z'}(\Delta)\cdot\overline{R}_1(n,z,\Delta)\\
&\cdot\frac{R_2((r+1)\Delta)}{R_2(r\Delta)}+(C_s+C_{P2}+C_{OP})\cdot\sum_{z'}P_{(z,n,r)(z',(n+1),1)}\\
&+(C_s+C_{F1}+C_{P2}+C_{OP}+C_{LP}T_F)\cdot\int_0^\Delta\big(1-\overline{R}_1(n,z,t)\big)\mathrm{d}t.
\end{aligned}$$

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